8.2 - Homogeneous Linear Systems

Consider the system $\begin{aligned}\frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y\end{aligned}$ which, in $\mathbf{X}' = \mathbf{A}\mathbf{X}$ form is

$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}$.

The solution is $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$.

This is a superposition of

$\vec{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ and $\vec{x}_2(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$

lambda

these have the form $\vec{x}(t) = \vec{K} e^{\lambda t}$

λ is the solution to an equation.

It is called an eigenvalue.

$\vec{K}$ is the associated eigenvector.

Suppose $\vec{x} = \vec{K} e^{\lambda t}$ is a solution to

$\vec{x}' = A\vec{x}$.

sub: $\vec{x}' = \lambda \vec{K} e^{\lambda t} \Rightarrow \lambda \vec{K} \cancel{e^{\lambda t}} = A \vec{K} \cancel{e^{\lambda t}}$

$\lambda \vec{K} = A \vec{K} \Rightarrow A\vec{K} - \lambda \vec{K} = \vec{0}$

$(A - \lambda I)\vec{K} = \vec{0}$

The zero vector

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

the identity matrix

One way to make $(A-\lambda I)\vec{K}=\vec{0}$ true is to have $\vec{K}=\vec{0}$. This is the trivial solution. A nontrivial solution is obtained when $\boxed{\det(A-\lambda I)=0}$. This is the characteristic equation.

from linear algebra

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \Rightarrow A-\lambda I = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$ (characteristic equation)

$$\lambda^2 - 5\lambda + \overset{+4}{6-2} = 0$$

$$(\lambda-4)(\lambda-1) = 0 \Rightarrow \lambda = 1, 4$$

$\lambda_1 = 1$: $(A - \lambda I)\vec{k} = \vec{0} \rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \vec{k} = 0$

$k_1 + 2k = 0 \Rightarrow k_1 = -2k_2$

$k_1 + 2k_2 = 0$

Let $k_2 = -1 \Rightarrow k_1 = 2 \Rightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda_2 = 4$: $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{aligned} -2k_1 + 2k_2 &= 0 \\ k_1 - k_2 &= 0 \end{aligned}$

Augmented matrix: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right)$

Want 0 here $\Rightarrow$

$R_2 \to R_1 + 2R_2$:
$$\begin{array}{ccc} -2 & 2 & 0 \\ 2 & -2 & 0 \\ \hline 0 & 0 & 0 \end{array}$$

$\downarrow$

$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$R_1 \to \frac{1}{2}R_1$ $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow -k_1 + k_2 = 0 \Rightarrow k_1 = k_2$

Let $k_1 = 1$. Then $\vec{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Recall that $\vec{X} = \vec{K} e^{\lambda t}$ so

$$\vec{X}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}, \quad \vec{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

so : $\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$

Aside: $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $A\vec{K} = \lambda \vec{K}$.

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 5x + 10y + 4z$$
$$\frac{dz}{dt} = 5y + 2z$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

$35(2-\lambda)$

$$(2-\lambda)(\lambda^2 - 12\lambda + 20 - 20) + 7(10 - 5\lambda) = 0$$

$$(2-\lambda)(\lambda^2 - 12\lambda + 35) = 0$$

$$(2-\lambda)(\lambda - 7)(\lambda - 5) = 0 \Rightarrow \lambda = 2, 5, 7$$

$\lambda_1 = 2$:

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}$$

$R_1 \Rightarrow 5R_1 + 7R_3$ then $R_3 \Rightarrow \frac{1}{5} R_3$

$$\begin{pmatrix} 0 & 0 & 0 \\ 5 & 8 & 4 \\ 0 & 1 & 0 \end{pmatrix}$$

$R_2 \Rightarrow R_2 - 8R_3$

$$\begin{array}{ccc|c} k_1 & k_2 & k_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$R_1 : 0 = 0 \quad R_3 : k_2 = 0$

$R_2 : 5k_1 + 4k_3 = 0 \Rightarrow k_1 = -\frac{4}{5} k_3$

Let $k_3 = 5$. Then $k_1 = -4$, and

$$\vec{K}_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} \qquad \vec{X}_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} e^{2t}$$

$\lambda_2 = 5$: $\begin{pmatrix} -3 & -7 & 0 \\ \boxed{5} & 5 & 4 \\ \boxed{0} & \boxed{5} & -3 \end{pmatrix}$

$R_2 \Rightarrow 5R_1 + 3R_2$

$$\begin{array}{ccc} -15 & -35 & 0 \\ 15 & 15 & 12 \\ \hline 0 & -20 & 12 \end{array}$$

Make these zeros first

make this zero second

Then $R_2 \Rightarrow \frac{1}{4} R_2$

$$\begin{pmatrix} -3 & -7 & 0 \\ 0 & -5 & 3 \\ 0 & 5 & -3 \end{pmatrix} \to \begin{pmatrix} -3 & -7 & 0 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

R_1: $-3k_1 = 7k_2 \Rightarrow k_1 = -\frac{7}{3}k_2$

R_2: $5k_2 = 3k_3 \Rightarrow k_3 = \frac{5}{3}k_2$

Let $k_2 = 3$. Then $k_1 = -7$, $k_3 = 5$

$$\vec{K}_2 = \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}$$

We can likewise find that for $\lambda = 7$,

$$\vec{K}_3 = \begin{pmatrix} 7 \\ -5 \\ -5 \end{pmatrix}$$

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix}-4\\0\\5\end{pmatrix}e^{2t} + c_2\begin{pmatrix}-7\\3\\5\end{pmatrix}e^{5t} + c_3\begin{pmatrix}7\\-5\\-5\end{pmatrix}e^{7t}}$$

$$\begin{vmatrix} \overset{+}{a} & b \\ \overset{-}{c} & d \end{vmatrix} = ad - bc$$

3x3 determinant:

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{vmatrix} = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$aei - afh$

$gec + \underline{afh} + bdi$

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$\underline{aei} + bfg + cdh$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\vec{X}' = A\vec{X}$ represents $\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x' = ax + by \\ y' = cx + dy \end{array}$

Back to the first example: (solution)

$$\vec{X}(t) = c_1\begin{pmatrix}2\\-1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$$

$$\vec{X} = c_1\vec{K}_1 e^{\lambda_1 t} + c_2\vec{K}_2 e^{\lambda_2 t} = \left(\vec{K}_1 e^{\lambda_1 t} \,\middle|\, \vec{K}_2 e^{\lambda_2 t}\right)\vec{C}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}}_{\vec{K}_1} + c_2 \underbrace{\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}}_{\vec{K}_2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2c_1 e^t + c_2 e^{4t} \\ -c_1 e^t + c_2 e^{4t} \end{pmatrix}$$

$x(t) = 2c_1 e^t + c_2 e^{4t}$

$y(t) = -c_1 e^t + c_2 e^{4t}$

$$\vec{X} = \begin{pmatrix} 2e^t & e^{4t} \\ -e^t & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Phi(t)\, \vec{C}$$

Phi (fee)

column vector of arbitrary constants

Fundamental matrix for the system

Complex eigenvalues:

If $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$, then we have

$\vec{X} = \vec{K} e^{\lambda t}$ and $\vec{X} = \bar{\vec{K}} e^{\bar{\lambda} t}$, where

$\bar{\lambda}$ is the complex conjugate of λ.

But $\vec{K} e^{(\alpha + \beta i)t} = \vec{K} e^{\alpha t}(\cos\beta t + i \sin\beta t)$,

$\bar{\vec{K}} e^{(\alpha - \beta i)t} = \bar{\vec{K}} e^{\alpha t}(\cos\beta t - i\sin\beta t)$

By the superposition principle,

$$\vec{X}_1 = \frac{1}{2}\left(\vec{K}e^{\lambda t} + \bar{\vec{K}}e^{\bar{\lambda} t}\right) = \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\cos\beta t - \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\sin\beta t$$

$$\vec{X}_2 = \frac{i}{2}\left(-\vec{K}e^{\lambda t} + \bar{\vec{K}}e^{\bar{\lambda} t}\right) = \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\cos\beta t + \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\sin\beta t$$

Aside: $\frac{1}{2}(z+\bar{z}) = \frac{1}{2}(\alpha+\beta i+\alpha-\beta i) = \alpha$

$$\frac{i}{2}(-z+\bar{z}) = \frac{i}{2}(-\alpha-\beta i+\alpha-\beta i) = \beta$$

So $\underline{\frac{1}{2}(z+\bar{z}) = \alpha = \mathrm{Re}(z)}$ (the real part)

$\underline{\frac{i}{2}(-z+\bar{z}) = \beta = \mathrm{Im}(z)}$ (the imaginary coeff)

So we let $\underline{\vec{B}_1 = \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right)}$ and

$$\underline{\vec{B}_2 = \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right)}$$

This leads to a theorem:

Theorem: Let $\lambda = \alpha + i\beta$ be an eigenvalue of $\mathbf{A}$ and let $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K})$, $\mathbf{B}_2 = \mathrm{Im}(\mathbf{K})$. Then

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t)$$
$$\mathbf{X}_2 = e^{\alpha t}(\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t)$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 4x + 5y$$
$$\frac{dy}{dt} = -2x + 6y$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + 24 + 10 = 0$$

$$\lambda^2 - 10\lambda + 25 = -34 + 25 \Rightarrow (\lambda - 5)^2 = -9$$

$$\lambda = 5 \pm 3i$$

$\lambda = 5 + 3i$: $\begin{pmatrix} 4-(5+3i) & 5 \\ -2 & 6-(5+3i) \end{pmatrix} \vec{K} = \vec{0}$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} \vec{K} = \vec{0}$$

Check λ is correct: $-(1+3i)(1-3i) + 10 = 0$ ✓

so if we row reduced, we'd have

$$\begin{pmatrix} -1-3i & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \vec{0} \Rightarrow (-1-3i)k_1 + 5k_2 = 0$$

$k_1 = \frac{5}{1+3i} k_2$ Let $k_2 = 1+3i \Rightarrow k_1 = 5$

$\vec{K} = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} i$ gives

$$\vec{B}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \quad \vec{B}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$\lambda = 5+3i$ WA: $\langle 5, 1\rangle \cos 3t$

$$\boxed{\vec{x} = c_1 e^{5t}\left[\begin{pmatrix}5\\1\end{pmatrix}\cos 3t - \begin{pmatrix}0\\3\end{pmatrix}\sin 3t\right] + c_2 e^{5t}\left[\begin{pmatrix}0\\3\end{pmatrix}\cos 3t + \begin{pmatrix}5\\1\end{pmatrix}\sin 3t\right]}$$

Meaning: $\vec{x} = c_1 e^{5t}\begin{pmatrix}5\cos 3t\\ \cos 3t - 3\sin 3t\end{pmatrix} + c_2 e^{5t}\begin{pmatrix}5\sin 3t\\ 3\cos 3t + \sin 3t\end{pmatrix}$

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}c_1\cos 3t + 5c_2\sin 3t\\ c_1\cos 3t - 3c_1\sin 3t + 3c_2\cos 3t + c_2\sin 3t\end{pmatrix}e^{5t}$$

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}c_1\cos 3t + 5c_2\sin 3t\\ (c_1+3c_2)\cos 3t + (-3c_1+c_2)\sin 3t\end{pmatrix}e^{5t}$$

$$\vec{x}(t) = \begin{pmatrix}5e^{5t}\cos 3t & 5e^{5t}\sin 3t\\ e^{5t}\cos 3t - 3e^{5t}\sin 3t & 3e^{5t}\cos 3t + e^{5t}\sin 3t\end{pmatrix}\vec{C}$$

$$\begin{array}{c}R_1\\R_2\end{array}\begin{pmatrix}-1-3i & 5\\ -2 & 1-3i\end{pmatrix} \quad R_2 \to 2R_1 + (-1-3i)R_2$$

$$\begin{array}{cc} -2-6i & 10\\ 2+6i & -10\\ \hline 0 & 0\end{array}$$

$-(1+3i)(1-3i) = -10$

Repeated eigenvalues:

Example: Find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \mathbf{X}$$

$$\begin{vmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 12\lambda + 36 = 0$$
$$(\lambda - 6)^2 = 0$$

$\lambda = 6$ (repeated, mult. 2)

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{K} = \vec{0} \qquad \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}$$

$(A - \lambda I)\vec{K} = \vec{0}$
$$\begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \vec{K} = \vec{0} \Rightarrow 2k_1 - 3k_2 = 0$$
$$\Rightarrow k_1 = \tfrac{3}{2} k_2$$

Let $k_2 = 2 \Rightarrow k_1 = 3 \Rightarrow \vec{K} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

In this case, we will seek a 2^{nd} vector $\vec{P}$: $(A - \lambda I)\vec{P} = \vec{K}$

Rationale will come later

(use $\vec{K}$ to find $\vec{P}$)

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \to \left(\begin{array}{cc|c} 6 & -9 & 3 \\ 4 & -6 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 2 & -3 & 1 \\ 2 & -3 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right) \quad 2p_1 - 3p_2 = 1$$

Let $p_2 = 0$

Then $p_1 = \frac{1}{2} \Rightarrow \vec{P} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$$\boxed{\vec{X}(t) = \underbrace{c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t}}_{\vec{X}_1} + \underbrace{c_2 \left[\overset{\vec{K}}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}} t e^{6t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{6t} \right]}_{\vec{X}_2}}$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 3x + 2y + 4z$$
$$\frac{dy}{dt} = 2x + 2z$$
$$\frac{dz}{dt} = 4x + 2y + 3z$$

We find $\lambda = 8, -1, -1$

$$\lambda_1 = 8 \rightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\lambda = -1: \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \vec{K} = \vec{0} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2k_1 + k_2 + 2k_3 = 0$$

If $k_2 = 0$, $k_1 = -k_3 \Rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

If $k_3 = 0$, $2k_1 = -k_2 \Rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

If $k_1 = 0$, $k_2 = -2k_3 \Rightarrow \cancel{\vec{K}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}}$$

Aforementioned rationale:

In the case of a repeated eigenvalue, we have

$$\vec{x}_1 = \vec{K} e^{\lambda t}$$

$$\vec{x}_2 = \vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t}$$

If mult. 2

$$\vec{x}_3 = \vec{K} \frac{1}{2!} t^2 e^{\lambda t} + P t e^{\lambda t} + Q e^{\lambda t}$$

If mult. 3

Suppose $\vec{x} = \vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t}$

$$\vec{x}\,' = \vec{K} e^{\lambda t} + \vec{K} \lambda t e^{\lambda t} + \vec{P} \lambda e^{\lambda t}$$

so $\vec{x}\,' = A\vec{x}$ becomes

$$K e^{\lambda t} + K \lambda t e^{\lambda t} + \vec{P} \lambda e^{\lambda t} = A(\vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t})$$

$$(A\vec{K} - \lambda \vec{K}) t e^{\lambda t} + (\vec{K} + \lambda \vec{P} - A\vec{P}) e^{\lambda t} = \vec{0}$$

$$A\vec{K} - \lambda \vec{K} = \vec{0} \qquad \vec{K} = A\vec{P} - \lambda \vec{P}$$

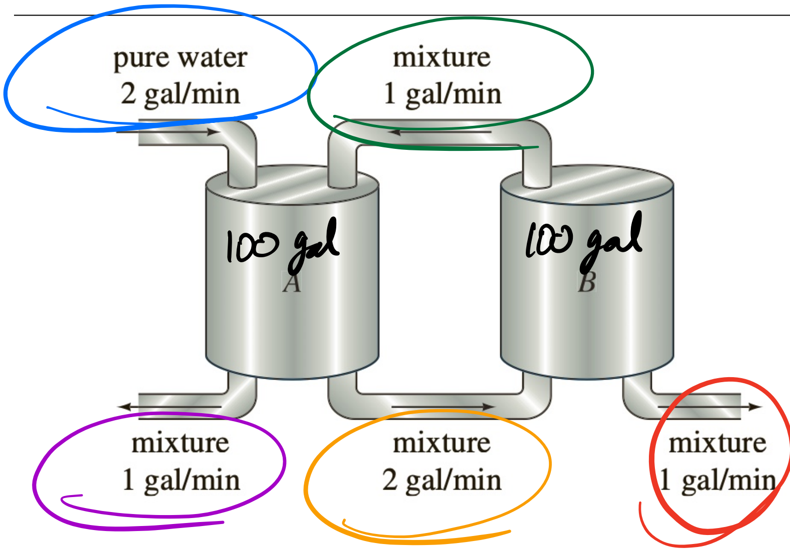
$$(A - \lambda I)\vec{K} = \vec{0} \qquad (A - \lambda I)\vec{P} = \vec{K}$$

we knew this

If Q is needed, we use

$$(A - \lambda I)\vec{Q} = \vec{P}$$

As shown in the diagram, two large connected mixing tanks A and B initially contain 100 gallons of brine. Liquid is pumped in and out of the tanks as indicated in the figure; the mixture pumped between and out of a tank is assumed to be well-stirred. Construct a mathematical model in the form of a linear system of first-order differential equations for the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t. Write the system in matrix form.



x_1 = # lb salt in A

x_2 = # lb salt in B

$$\frac{dx}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$\frac{dx_1}{dt} = 0 + \frac{x_2}{100} - \frac{x_1}{100} - \frac{2x_1}{100} = -\frac{3x_1}{100} + \frac{x_2}{100}$$

$$\frac{dx_2}{dt} = \frac{2x_1}{100} - \frac{x_2}{100} - \frac{x_2}{100} = \frac{x_1}{50} - \frac{x_2}{50}$$

$$X' = \begin{pmatrix} -3/100 & 1/100 \\ 1/50 & -1/50 \end{pmatrix} \vec{X}$$

AI response to where this model is used: This type of coupled mixing tank model shows up in several real engineering and scientific contexts:

Environmental & Water Treatment The most direct application is **wastewater treatment plants**, where tanks are linked in series or loops to progressively dilute or react contaminants. The exact flow-in/flow-out balance you see here models how pollutant concentrations evolve through each stage.

Pharmacokinetics (Drug Modeling) This is probably the *most important* real use. The human body is modeled as a series of "compartments" — blood plasma, tissue, organs — and a drug moves between them at known rates. The math is identical to these tank equations. Pharmacologists use this to predict:

- How long a drug stays in your system
- When blood concentration peaks
- Proper dosing intervals
- Chemical Reactors

In industrial chemistry, **continuous stirred-tank reactors (CSTRs)** are literally this setup. Reactants flow in, products flow out, and intermediate products recirculate between vessels. Petroleum refining and pharmaceutical manufacturing use multi-tank CSTR networks constantly.

Groundwater Contamination Aquifers are modeled as interconnected "cells" exchanging water at known rates. When a contaminant spills, hydrologists use these equations to predict how the plume spreads between underground zones over time.

HVAC and Atmospheric Modeling The atmosphere itself is divided into compartments (troposphere, boundary layer, urban airshed) with pollutants like CO_2 or smog transferring between them — same mathematical structure.

The reason this model is so powerful is that **any system where a substance moves between well-mixed reservoirs at proportional rates** reduces to the same form of coupled linear ODEs. The tanks are just the most intuitive way to introduce the concept.